

## Definitions

$T$	= target
$LSL$	= lower specification limit
$USL$	= upper specification limit
$\mu$	= process mean
$tol$	= sigma tolerance
$m$	= midpoint of LSL and USL
$\bar{x}$	= estimate of process mean
$\sigma_{Within}$	= within subgroup process standard deviation
$\hat{\sigma}_{Within} = \text{StDev(Within)}$	= estimate of within subgroup process standard deviation
$n$	= total number of (nonmissing) observations
$n_i$	= number of (nonmissing) observations in subgroup $i$
$c_4(n_i)$	= unbiasing constant for subgroups of size $n_i$ (for use with sample standard deviations)
$\sigma_{Overall}$	= overall process standard deviation
$\hat{\sigma}_{Overall} = \text{StDev(Overall)}$	= estimate of overall process standard deviation
$P(X)$	= probability of event $X$
$Z$	= standard normal variable
$x_i$	= the $i$ th observation
$x_{ij}$	= the $j$ th observation of the $i$ th subgroup

Note: If the subgroup size is 1,  $\hat{\sigma}_{Within}$  is estimated using the average moving range by default. If the subgroup size is greater than 1,  $\hat{\sigma}_{Within}$  is estimated using a pooled standard deviation by default. See Methods of Estimating Sigma below.

Note:  $c_4(n_i)$  is an unbiasing constant whether or not the sample size is small or large. The bias is negligible for large samples (see, e.g. Farnum), as with larger samples,  $c_4(n_i)$  approaches the value of one.

In Releases 13 and 14 you can estimate sigma either using unbiasing constants or not. The default is to use unbiasing constants. To not use unbiasing constants, choose Stat > Quality Tools > Capability Analysis > Normal > Estimate (In Release 13, Stat > Quality Tools > Capability Analysis (Normal) > Estimate) and uncheck Use unbiasing constants.

## Formulas

### Capability Statistics (Cp, Pp)

$$Cp = \frac{(USL - LSL)}{(6 * \sigma_{Within})}$$

$$\hat{C}p = \frac{(USL - LSL)}{(6 * \hat{\sigma}_{Within})}$$

$$CPU = \frac{(USL - \mu)}{(3 * \sigma_{Within})}$$

$$C\hat{P}U = \frac{(USL - \bar{x})}{(3 * \hat{\sigma}_{Within})}$$

$$CPL = \frac{(\mu - LSL)}{(3 * \sigma_{Within})}$$

$$C\hat{P}L = \frac{(\bar{x} - LSL)}{(3 * \hat{\sigma}_{Within})}$$

$$Cpk = \text{minimum}\{CPU, CPL\}$$

$$C\hat{p}k = \text{minimum}\{C\hat{P}U, C\hat{P}L\}$$

$\hat{C}_{pm} = \frac{(USL - LSL)}{tol * \sqrt{\frac{\sum (x_i - T)^2}{(n-1)}}}$	when LSL and USL are given, and T=m	Note: This formula is always used in Release 13. Cpm is not calculated in Release 13 if there is only one specification limit.
$\hat{C}_{pm} = \frac{\min(T - LSL, USL - T)}{\frac{tol}{2} * \sqrt{\frac{\sum (x_i - T)^2}{(n-1)}}}$	when LSL and USL are given, and T≠m	Note: This formula only applies to Release 14.
$\hat{C}_{pm} = \frac{(USL - T)}{\frac{tol}{2} * \sqrt{\frac{\sum (x_i - T)^2}{(n-1)}}}$	when USL and T are given	Note: This formula only applies to Release 14.
$\hat{C}_{pm} = \frac{(T - LSL)}{\frac{tol}{2} * \sqrt{\frac{\sum (x_i - T)^2}{(n-1)}}}$	when LSL and T are given	Note: This formula only applies to Release 14.

$C\hat{C}pk = \frac{USL - \hat{\mu}}{3\hat{\sigma}_{within}}$	when only USL is given
$C\hat{C}pk = \frac{\hat{\mu} - LSL}{3\hat{\sigma}_{within}}$	when only LSL is given
$C\hat{C}pk = \frac{\min\{(USL - \hat{\mu}), (\hat{\mu} - LSL)\}}{3\hat{\sigma}_{within}}$	when LSL and USL are given

where:

- $\hat{\mu}$  = Target                      when Target is given  
 $\hat{\mu} = \frac{1}{2}(LSL + USL)$             when both LSL and USL are given but no Target  
 $\hat{\mu} = \bar{x}$                                 otherwise

$$Pp = \frac{(USL - LSL)}{6 * \sigma_{Overall}}$$

$$\hat{P}p = \frac{(USL - LSL)}{6 * \hat{\sigma}_{Overall}}$$

$$PPU = \frac{(USL - \mu)}{3 * \sigma_{Overall}}$$

$$P\hat{P}U = \frac{(USL - \bar{x})}{(3 * \hat{\sigma}_{Overall})}$$

$$PPL = \frac{(\mu - LSL)}{3 * \sigma_{Overall}}$$

$$P\hat{P}L = \frac{(\bar{x} - LSL)}{(3 * \hat{\sigma}_{Overall})}$$

$$Ppk = \text{minimum}\{PPU, PPL\}$$

$$P\hat{p}k = \text{minimum}\{P\hat{P}U, P\hat{P}L\}$$

Note: The capability statistics displayed in the MINITAB output are estimates, but for simplicity do not include the “hats”. For example,  $C\hat{p}k$  is displayed as Cpk.

Note: In Release 14, you can display confidence intervals for the capability statistics. To see the formulas, see Confidence Intervals below.

## Confidence Intervals for Capability Statistics

### Definitions for Confidence Intervals

$$\nu = f_n * df$$

$$df = k(n-1)$$

k = number of samples

n = average sample size

$f_n$  = adjustment for method used to estimate  $\sigma_{within}^2$

For:

- Pooled standard deviation,  $f_n = 1$
- Average and median moving range,  $\nu = k - w + 1$  where w is the number of observations used to calculate the moving range
- Sbar,  $f_n$  varies with n

n	$f_n$
2	0.88
3	0.92
4	0.94
5	0.95
6-7	0.96
8-9	0.97
10-	0.98
17	
18-	0.99
64	
65+	1.00

- Rbar,  $f_n = 0.9$
- Square root of MSSD,  $\nu = k - 1$

**Formulas for Confidence Intervals**

Cp  

$$LowerBound = \hat{C}p \sqrt{\frac{\chi^2_{1-\alpha/2, \nu}}{\nu}}, UpperBound = \hat{C}p \sqrt{\frac{\chi^2_{\alpha/2, \nu}}{\nu}}$$

where  $\nu$  is defined under “Definitions of Confidence Intervals” above.

Cpk  

$$LowerBound = \hat{C}pk - Z_{\alpha/2} \sqrt{\frac{1}{9kn} + \frac{\hat{C}pk^2}{2\nu}}, UpperBound = \hat{C}pk + Z_{\alpha/2} \sqrt{\frac{1}{9kn} + \frac{\hat{C}pk^2}{2\nu}}$$

where  $k$ ,  $n$ , and  $\nu$  are defined under “Definitions of Confidence Intervals” above.

Cpm  

$$LowerBound = \hat{C}pm \sqrt{\frac{\chi^2_{1-\alpha/2, \nu}}{\nu}}, UpperBound = \hat{C}pm \sqrt{\frac{\chi^2_{\alpha/2, \nu}}{\nu}}$$

where:

$$\nu = kn \frac{(1 + a^2)^2}{1 + 2a^2}$$

$$a = \frac{\bar{x} - Target}{\hat{\sigma}_{overall}}$$

$k$  and  $n$  are defined under “Definitions of Confidence Intervals” above

Pp  

$$LowerBound = P\hat{p} \sqrt{\frac{\chi^2_{1-\alpha/2, kn-1}}{kn-1}}, UpperBound = P\hat{p} \sqrt{\frac{\chi^2_{\alpha/2, kn-1}}{kn-1}}$$

where  $k$  and  $n$  are defined under “Definitions of Confidence Intervals” above.

Ppk  

$$LowerBound = P\hat{p}k - Z_{\alpha/2} \sqrt{\frac{1}{9kn} + \frac{P\hat{p}k^2}{2(kn-1)}}, UpperBound = P\hat{p}k + Z_{\alpha/2} \sqrt{\frac{1}{9kn} + \frac{P\hat{p}k^2}{2(kn-1)}}$$

where  $k$  and  $n$  are defined under “Definitions of Confidence Intervals” above.

## PPM

i) Observed Performance PPM  $< LSL = 1,000,000 * ((\text{number of observations} < LSL) / n)$

ii) Observed Performance PPM  $> USL = 1,000,000 * ((\text{number of observations} > USL) / n)$

Observed Performance Total = sum of (i) and (ii)

iii) Exp. “Within” Performance PPM  $< LSL = 1,000,000 * P\left(Z > \frac{(LSL - \bar{x})}{\hat{\sigma}_{Within}}\right)$

iv) Exp. “Within” Performance PPM  $> USL = 1,000,000 * P\left(Z > \frac{(USL - \bar{x})}{\hat{\sigma}_{Within}}\right)$

Exp. “Within” Performance Total = sum of (iii) and (iv)

v) Exp. “Overall” Performance PPM  $< LSL = 1,000,000 * P\left(Z > \frac{(LSL - \bar{x})}{\hat{\sigma}_{Overall}}\right)$

vi) Exp. “Overall” Performance PPM  $> USL = 1,000,000 * P\left(Z > \frac{(USL - \bar{x})}{\hat{\sigma}_{Overall}}\right)$

Exp. “Overall” Performance Total = sum of (v) and (vi)

Note: % figures are obtained by dividing the formulas above by 10,000.

Note: If you use a historical mean, that value will be used in place of  $\bar{X}$  for  $\hat{C}_{PU}$ ,  $\hat{C}_{PL}$ ,  $\hat{C}_{pk}$ ,  $\hat{P}_{PU}$ ,  $\hat{P}_{PL}$ ,  $\hat{P}_{pk}$ , and all PPM (Expected).

If you use a historical sigma, that value will be used in place of  $\hat{\sigma}_{Within}$  for  $\hat{C}_p$ ,  $\hat{C}_{PU}$ ,  $\hat{C}_{PL}$ , and all PPM (Expected “Within”).

## Methods of Estimating $\sigma$

### Overall standard deviation

$$\hat{\sigma}_{Overall} = \text{StDev(Overall)} = \frac{s}{c_4(n)}$$

$$s = \sqrt{\frac{\sum_i \sum_j (x_{ij} - \bar{x})^2}{(n-1)}}$$

### Estimating $\sigma_{Within}$ when subgroup size > 1

#### Pooled standard deviation – Default

$$\hat{\sigma}_{Within} = \text{StDev(Within)} = \frac{s_p}{c_4(d)}$$

$$s_p = \sqrt{\frac{\sum_i \sum_j (x_{ij} - \bar{x}_i)^2}{\sum_i (n_i - 1)}}$$

$$d = \sum (n_i - 1) + 1 \text{ degrees of freedom}$$

Note: When the subgroup size is constant, the pooled standard deviation = (square root of the average of the variances) /  $c_4(\sum(n_i-1)+1)$ :

$$\hat{\sigma}_{Within} = \text{StDev(Within)} = \frac{s_p}{c_4(d)}$$

$$s_p = \sqrt{\frac{\sum_{i=1}^k s_i^2}{k}}$$

$$d = n - k + 1 \text{ degrees of freedom}$$

k = number of subgroups

Rbar - Average of subgroup ranges

$$\hat{\sigma}_{Within} = \text{StDev(Within)} = \frac{\sum_i \left( \frac{f_i r_i}{d_2(n_i)} \right)}{\sum_i f_i}$$

$$f_i = \frac{[d_2(n)]^2}{[d_3(n)]^2}$$

$$\hat{\sigma}_{Within} = \text{StDev(Within)} = \frac{\bar{R}}{d_2(n_i)} \text{ if all } n_i \text{ are the same}$$

**Sbar - Average of subgroup standard deviations**

$$\hat{\sigma}_{Within} = \text{StDev(Within)} = \frac{\sum \left( \frac{h_i s_i}{c_4(n_i)} \right)}{\sum (h_i)}$$

$$s_i = \sqrt{\frac{\sum_j (x_{ij} - \bar{x}_i)^2}{(n_i - 1)}}$$

$$h_i = \frac{(c_4(n_i))^2}{1 - (c_4(n_i))^2}$$

Note: When the subgroup size is constant, sbar = (average of the subgroup standard deviations) / c<sub>4</sub>(n<sub>i</sub>):

$$\hat{\sigma}_{Within} = \text{StDev(Within)} = \frac{\bar{s}}{c_4(n_i)}$$

$$\bar{s} = \frac{\sum_{i=1}^k s_i}{k}$$

k = number of subgroups



**Estimating  $\sigma_{\text{Within}}$  when subgroup size = 1**

**Average moving range – Default**

$$\hat{\sigma}_{\text{Within}} = \text{StDev}(\text{Within}) = \frac{\overline{MR}}{d_2(w)}$$

$MR_i$  = the  $i$ th moving range =  $\max[x_i, \dots, x_{i-w+1}] - \min[x_i, \dots, x_{i-w+1}]$ , for  $i = w, \dots, n$

$$\overline{MR} = \frac{(MR_w + \dots + MR_n)}{(n - w + 1)}$$

$w$  = the number of observations used in the moving range.  $w = 2$  by default.

**Median moving range**

$$\hat{\sigma}_{\text{Within}} = \text{StDev}(\text{Within}) = \frac{\tilde{MR}}{d_4(w)}$$

$MR_i$  = the  $i$ th moving range =  $\max[x_i, \dots, x_{i-w+1}] - \min[x_i, \dots, x_{i-w+1}]$ , for  $i = w, \dots, n$

$\tilde{MR}$  = the median of all  $MR_i$

$w$  = the number of observation used in the moving range.  $w = 2$  by default.

**Square root of MSSD – Square root of half of the Mean of the Squared Successive Differences**

$$\hat{\sigma}_{\text{Within}} = \text{StDev}(\text{Within}) = \frac{\sqrt{\frac{1}{2} * \frac{(\sum d_i^2)}{(n-1)}}}{c_4(n_i)}$$

$d_i$  = successive differences

$c_4(n_i)' \cong c_4(n_i)$

**References:**

- Montgomery, D.C. (2001). Introduction to Statistical Quality Control, 4<sup>th</sup> Edition. John Wiley & Sons.  
 Donald J. Wheeler(1995). "Advanced Topics in Statistical Process Control", SPC Press, Inc.  
 N. R. Farnum (1994). "Modern Statistical Quality Control and Improvement", Wadsworth Publishing.  
 I.W. Burr (1976). Statistical Quality Control Methods, Marcel Dekker, Inc.