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THE KOLMOGOROFF METHOD

If in a computer we have the coefficients (x_0, x_1, x_2, \dots) of a polynomial $X(Z)$, we say we are "working in the time domain." If we evaluate the polynomial $X(Z)$ at a number of positions on the unit circle, we have numbers, say $X(e^{i\omega_0}), X(e^{i\omega_1}), \dots, X(e^{i\omega_n})$, which we call a frequency-domain representation of the polynomial $X(Z)$. We have seen that the fast Fourier transform is a very cheap way of going from the time domain to the frequency domain and back. This makes it very worthwhile to look at Whittle's factorization method in the frequency domain. Furthermore, we will understand spectral factorization from yet another point of view.

We may begin with a time function or Z transform $X(Z) = x_0 + x_1 Z + \dots$. Let us denote by X_k the transform of the time function, that is, $X(Z)$ evaluated at numerous $(k = 0, 1, \dots, n)$ places on the unit circle. Consider the identities

$$R_k = \bar{X}_k X_k = e^{i\Phi_k} = e^{j\theta_k}$$

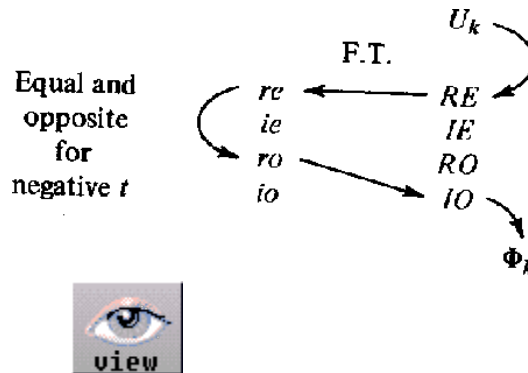
Now we add and subtract a still arbitrary function Φ_k to the exponential

$$R_k = \exp\left[\frac{1}{2}(U_k - i\Phi_k)\right] \exp\left[\frac{1}{2}(U_k + i\Phi_k)\right] = \bar{B}_k B_k$$

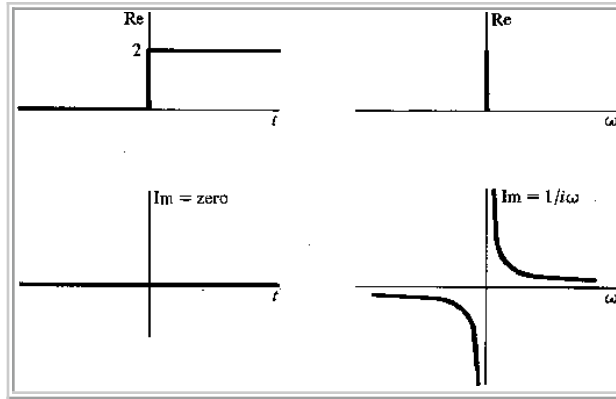
Now the big question is what Φ_k should be used to guarantee that B_k transforms to a minimum-phase, one-sided time function? By looking at Whittle's method, we note that the only significant properties of $U^+(Z)$ are that it is finite and that the time function u_t vanishes before $t = 0$. Thus we expect that Φ_k should be chosen so that when $U_k + i\Phi_k$ is transformed into the time domain the resulting time function u^+_t should vanish for negative time. This may be done as depicted in Figure 4.

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Figure 4 Determination of the phase function.



To see how easy it really is to get the imaginary odd part IO , we fetch the integration filter from Sec. 2-8 (on bilinear transformation) and display it in Figure 5.



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Figure 5 The transform pair used in the Hilbert transform.



To get Φ_k we take U_k into the time domain, getting u_t . Then we multiply by the real step function of time in Figure 5, obtaining $x_t^+ = x_0, x_1, \dots$. This implies that in the frequency domain U_k has been convolved with $\delta_{\omega=0} + i \times (90^\circ \text{ phase shift filter})$. Thus, Φ has been generated.

Let us reconsider the operation of dropping all of the negative powers of Z in $U(Z)$ as we did in the previous section to get $U^+(Z)$. For simplicity, consider the case r_t real; then u_t is real.

$$U(Z) = \dots + x_1 Z^{-1} + x_0 + x_1 Z + x_2 Z^2 + \dots$$

$$U = x_0(\cos 0) + 2x_1 \cos \omega + 2x_2 \cos 2\omega + \dots$$

Now let us make up a new function Φ by replacing cosine by sine in the foregoing expression

$$\Phi = 2x_1 \sin \omega + 2x_2 \sin 2\omega + \dots$$

We now see that combining U with $i\Phi$ we get U^+ .

$$\frac{1}{2}(U + i\Phi) = \frac{1}{2}x_0 + x_1 Z + x_2 Z^2 + \dots$$

$$= U^+(Z)$$

Notice that the operation of changing $\cos t$ to $\sin t$ would be called 90° phase shift filtering. Here we have changed $\cos \omega$ to $\sin \omega$ with the result that $U^+(Z)$ has only positive coefficients of Z .

The Kolmogoroff method of spectral factorization is very fast in a computer because fast Fourier transforms may be used. Its principle disadvantage is that summation around the unit circle is always slightly different than integration about the circle. When the spectrum is simple but poles are very close to the unit circle, then the Toeplitz method may prove more satisfactory. A simple program to do spectral factorization is given in Figure 6.

EXERCISES:

1. Insert the additional arrows in Figure 4 which are required when dealing with complex time functions.
2. What is the meaning of *minimum-phase waveform* if the roles of time domain and frequency domain are interchanged?

3. Show how to do the inverse Hilbert transform, given ϕ find u . What is the interpretation of the fact that you cannot get u_0 ?
4. Consider a model of a portion of the earth where x is the north coordinate, $+z$ represents altitude above the earth, and magnetic bodies are distributed in the earth so as to create no magnetic field component in the east-west direction. One may show that the magnetic field h above the earth is represented by

$$\begin{bmatrix} h_x(x, z) \\ h_z(x, z) \end{bmatrix} = \int_{-\infty}^{+\infty} F(k) \begin{bmatrix} -ik \\ |k| \end{bmatrix} e^{iks - |k|z} dk$$

Here $F(k)$ is some spatial frequency spectrum.

- (a) By using Fourier transforms, how does one compute $h_x(x, 0)$ from $h_z(x, 0)$ and vice versa?
- (b) Given $h_z(x, 0)$, how does one compute $h_z(x, z)$?
- (c) Notice that at $z = 0$

$$u(x) = h_x(x) + ih_z(x) = \int_{-\infty}^{+\infty} e^{iks} F(k) (|k| + k) dk$$

and that $F(k) (|k| + k)$ is a one-sided function of k . With a total field magnetometer one observes

$$h_x^2(x) + h_z^2(x) = u(x)\bar{u}(x)$$

What can you say about getting $F(k)$ from this?

- (d) How unique are $h_x(x)$ and $h_z(x)$ if $u(x)\bar{u}(x)$ is given?

<pre> LX=4 WRITE(6,FMT) (CX(I),I=1,LX) CALL FORK(LX,CX,1.) DO 10 I=1,LX 10 CX(I)=.5*CLOG(CX(I)*CONJG(CX(I))) CALL FORK(LX,CX,-1.) K=LX/2 DO 20 J=2,K CX(I)=CX(J)+CX(J) 20 CX(LX+2-J)=0. CALL FORK(LX,CX,+1.) DO 30 I=1,LX 30 CX(I)=CEXP(CX(I)) CALL FORK(LX,CX,-1.) WRITE(6,FMT) (CX(I),I=1,LX) </pre>	<pre> LX MUST BE A POWER OF 2. PRINT THE TEST EXAMPLE. FOURIER TRANSFORM. FORM THE LOG OF THE SPECTRUM BEGIN THE HILBERT TRANSFORM LX MUST BE EVEN LEAVE I=0 ALONE DOUBLE VALUES AT POSITIVE TIME ZERO VALUES AT NEGATIVE TIME END HILBERT TRANSFORM EXPONENTIATE INVERSE FOURIER TRANSFORM PRINT MINIMUM PHASE WAVELET. </pre>
<pre> IN: (0.0, 0.0), (0.0, 8.0), (16.0, 0.0), (0.0, 0.0), OUT: (15.6,-0.0), (0.0,-8.7), (-0.4,-0.0), (0.0,-0.7), </pre>	

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Figure 6 A program to do spectral factorization by means of fast Fourier transform and Hilbert transform. Complex arithmetic is mandatory. Results are approximate since integration around the unit

circle has been approximated by summation over four points.



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